LP Infeasibility - The Pursuit of Intelligent Discretization

WORKSHOP ON CONVEX AND REAL-TIME OPTIMIZATION

August 19, 2016

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Agenda

Introduction

Bernstein Basis

Polynomials in the Bernstein Basis

Lyapunov Functions in the Bernstein Basis

Example

Infeasibility

Example Continued

Constraint Identification

Farkas' Lemma

Open Questions

Lyapunov Fcn for Stability Analysis

Classical Lyapunov criteria:

$$V(x) > 0 \wedge V(0) = 0,$$
$$\dot{V}(x) \leq 0.$$

Thus V is positive definite (PD) and $-\dot{V}$ is either PS or positive semi-definite (PSD).

Lyapunov Fcn for Stability Analysis

Classical Lyapunov criteria:

$$V(x) > 0 \wedge V(0) = 0,$$
$$\dot{V}(x) \leq 0.$$

Current standard is to relax as

$$V(x) \in \Sigma^2$$
$$-\dot{V}(x) \in \Sigma^2$$

for global analysis, or as

$$V(x) - \Sigma_i s_i p_i \in \Sigma^2$$
$$-\dot{V}(x) - \Sigma_i s_i p_i \in \Sigma^2$$

for local analysis using Putinar's Positivestellensatz.

Another Way

But PSD $\neq$ SOS. And solving the SOS programs require semi-definite programming (SDP).

We do something else. Our method relies on the Bernstein basis. It always results in local analysis, but is also always results in linear programming (LP).

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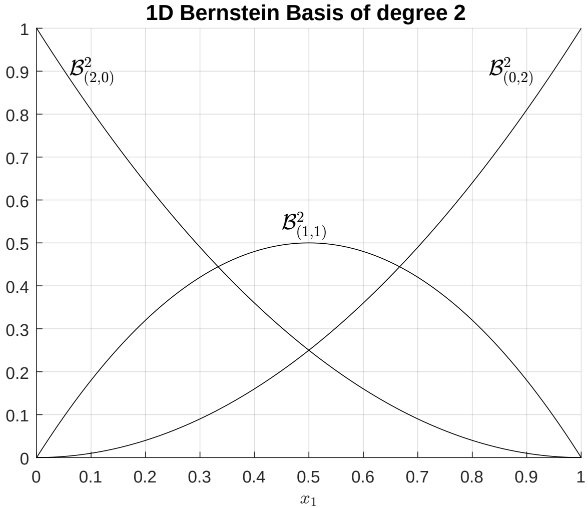
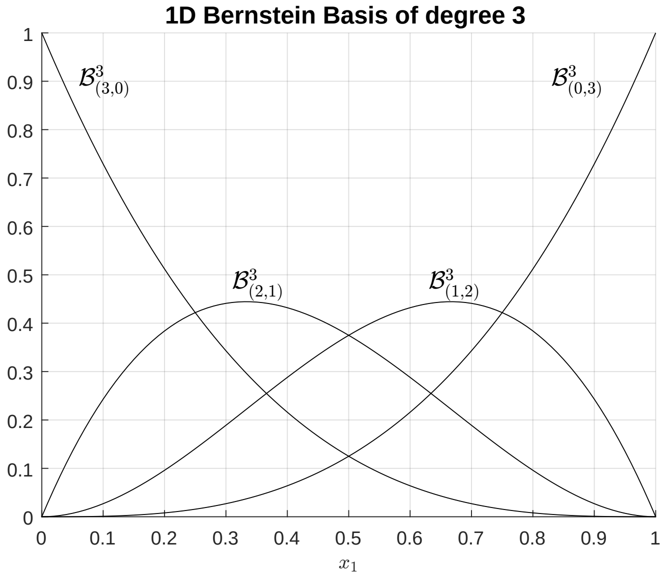
Bernstein basis Polynomials

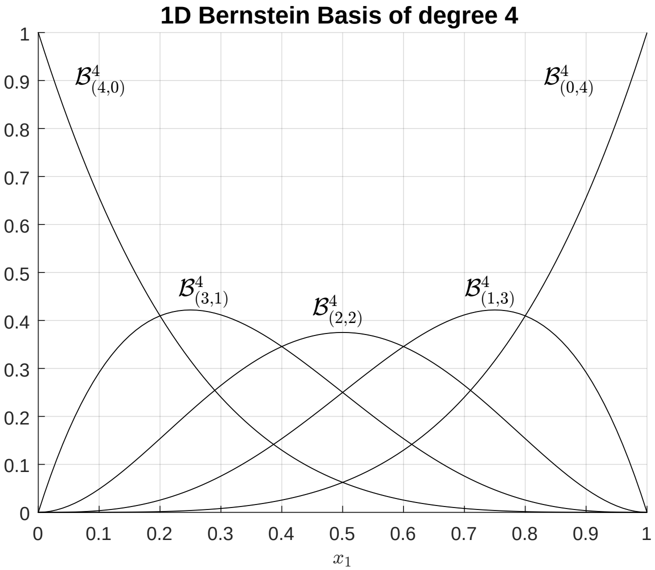
Defined on a simplex σ: 1D is a line, 2D is a triangle, 3D and up is called a simplex.



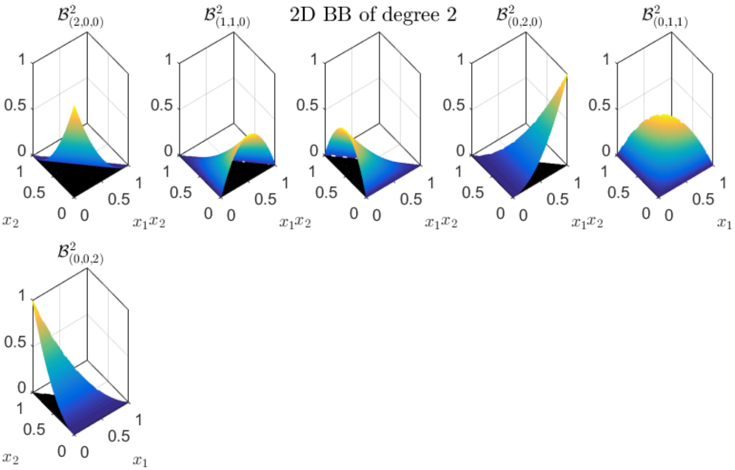
Bernstein basis Polynomials



Bernstein basis Polynomials



Bernstein basis Polynomials



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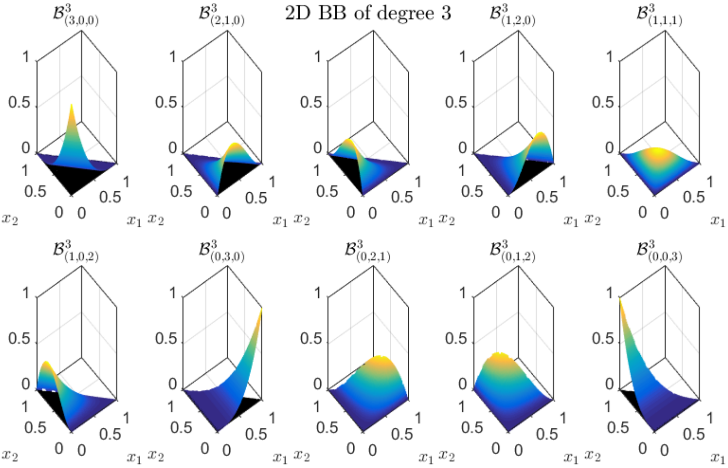
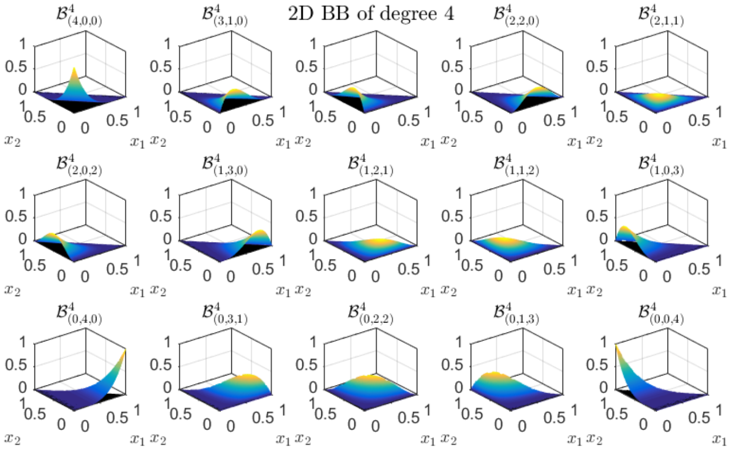
Bernstein basis Polynomials



Bernstein basis Polynomials



Bernstein Coefficients

Any polynomial can be described in the Bernstein basis by a linear combination of the Bernstein basis polynomials, $\mathcal{B}_\alpha^d$.

$$p = \sum_{|\alpha|=d} b_\alpha(p, d, \sigma)\mathcal{B}_\alpha^d$$

The Bernstein coefficients contain valuable information about the polynomial p. The basis polynomials are non-negative on the simplex. Thus, if the coefficients are positive then so is p on the simplex.

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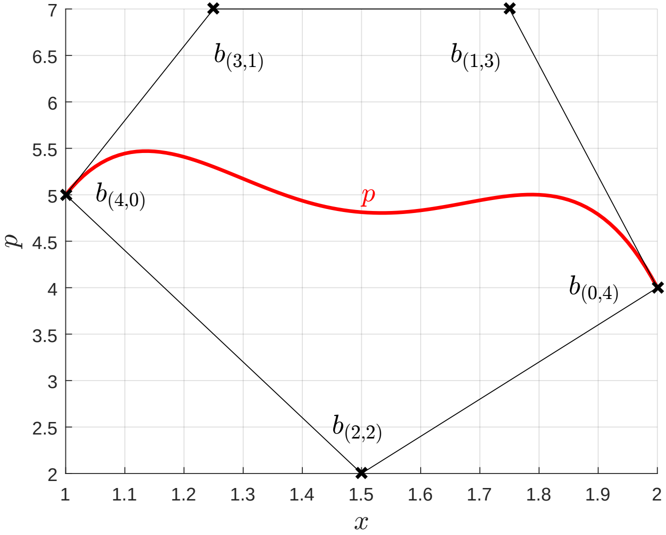
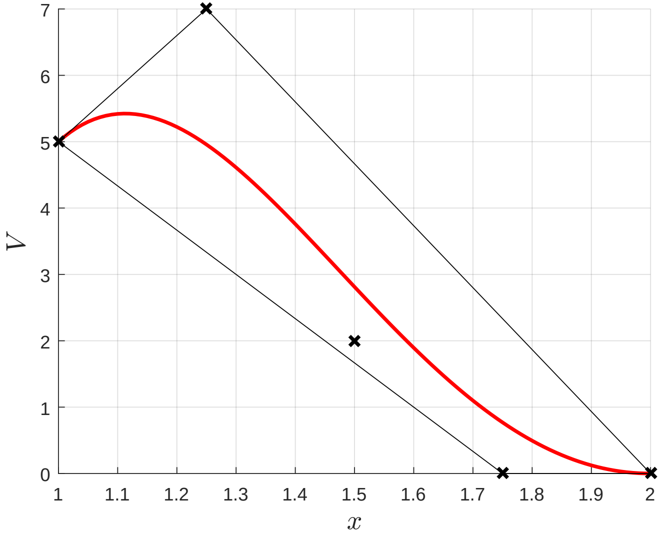
Convex Hull & End-point Value Property



Classical Lyapunov criteria



Now to the Fun of it!

$$V = \sum_{|\alpha_V| = d_V} CV_{\alpha_V} \mathcal{B}^{d_V}_{\alpha_V} = CV^T \mathcal{B}^{d_V}$$

$$\dot{V} = \sum_{|\hat{\alpha}| = \hat{d}} CL_{\hat{\alpha}} \mathcal{B}^{\hat{d}}_{\hat{\alpha}} = CL^T \mathcal{B}^{\hat{d}}$$

$$CV \geq 0 \qquad CV_{\alpha_V^*} = 0$$

$$CL \leq 0 \qquad CL_{\hat{\alpha}^*} = 0$$

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Now to the Fun of it!

$$\begin{aligned} V &= \sum_{|\alpha_V|=d_V} CV_{\alpha_V}\mathcal{B}^{d_V}_{\alpha_V} = CV^T\mathcal{B}^{d_V} \\ \dot{V} &= \sum_{|\hat{\alpha}|=\hat{d}} CL_{\hat{\alpha}}\mathcal{B}^{\hat{d}}_{\hat{\alpha}} = CL^T\mathcal{B}^{\hat{d}} \end{aligned}$$

$$CV \geq 0 \qquad CV_{\alpha_V^*} = 0$$

$$CL \leq 0 \qquad CL_{\hat{\alpha}^*} = 0$$

$$CL = A\,CV$$

Linear Feasibility Problem

$$\begin{aligned} \min_{CV} \quad & 0 \\ s.t. \quad & l^c \leq A\,CV \leq 0 \\ & 0 \leq CV \leq u^x \end{aligned}$$

Any CV solving the LP problem is a Lyapunov function for the vector field and certifies the local stability. Infeasibility does not suggest the converse!

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The Vector Field

$$\dot{x}_1 = -x_1^3x_2^2 + 2x_1^3x_2 - x_1^3 + 4x_1^2x_2^2 - 8x_1^2x_2 + 4x_1^2 - x_1x_2^4 + 4x_1x_2^3 - 4x_1 + 10x_2^2$$

$$\dot{x}_2 = -9x_1^2x_2 + 10x_1^2 + 2x_1x_2^3 - 8x_1x_2^2 - 4x_1 - x_2^3 + 4x_2^2 - 4x_2$$

Two variables, degree 5, origin is stable and it is the only equilibrium.
Investigated on the box $B = [\pm 1]^2$ with a Lyapunov function of degree $d_V = 2$.

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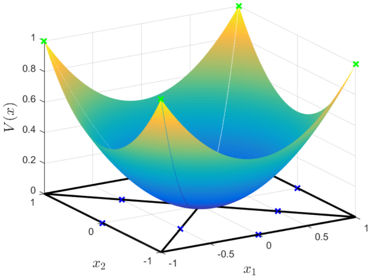
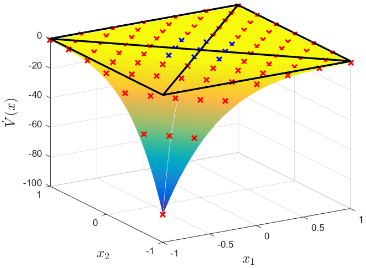
The Solution





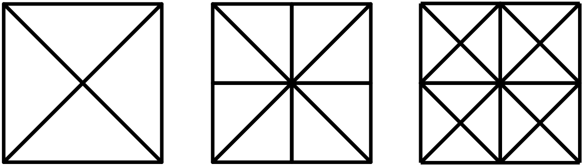
Infeasibility

But if the box is expanded to $B = [\pm 1.75]^2$ the resulting LP is unsolvable. Then what?

Bernstein's Theorem

Bernstein's Theorem: If a polynomial p of degree d is positive on a simplex σ, then there exists a sub-division of σ into a collection $K = \{\sigma^1, \cdots, \sigma^m\}$ of finitely many simplices such that

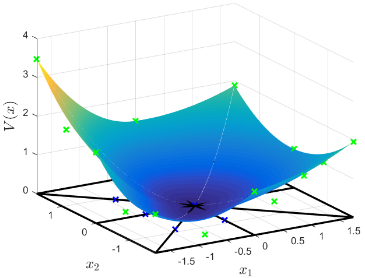
$$b(p, d, \sigma^i) > 0 \forall i \in \{1, \cdots, m\}.$$

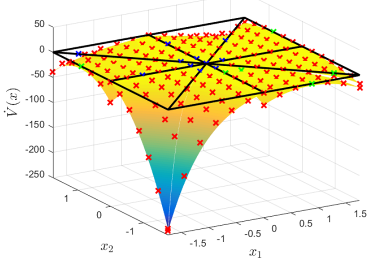


This motivates the use of sub-division in order to obtain a solvable LP.

The Solution

Sub-dividing into 8 simplices renders the LP solvable.





But the number of design variables and constraints grow rapidly with increase in the number of simplices.

Adding Slack Variables

By adding one non-negative slack variable on each simplex, and minimizing their sum, the problematic parts of the domain can be identified

$$\min_{CV,s_i} \sum_{i}^{m} s_i$$
$$s.t. \quad l^c \leq A\,CV - Bs \leq 0$$
$$0 \leq CV \leq u^x$$
$$0 \leq s_i$$

This singles out some simplices for further sub-division, and leaves the rest as they are.

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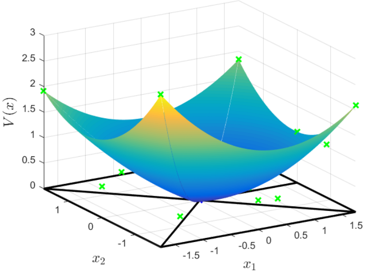
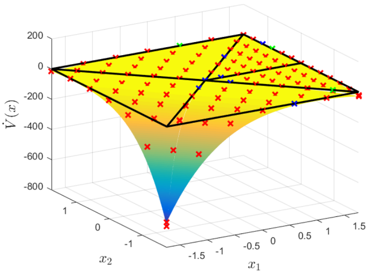
The Solution

And it works.





Saves on the number of design variables and constraints, yet obtains a solvable LP.

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Advertisement

Everything so far is covered in the paper:

T. Leth, C. Sloth, and R. Wisniewski, "Lyapunov Function Synthesis - Algorithm and Software," *MSC*, *Buenos Aries*, *Argentina*, 2016.

New Stuff

Since writing the paper, Farkas' Lemma has gained my attention as the constraint identifying mechanism.

Farkas' Lemma (simplified): Given A and b defining an LP, exactly one of the two proposition it true:

1. $\exists x \ : \ Ax = b, \ x \geq 0,$
2. $\exists y \ : \ A^T y \leq 0, \ b^T y > 0.$

If such a y exists it is a certificate of infeasibility of 1. More importantly, the non-zero entries of y implies variables and constraints which are important for the infeasibility.

Two important features:

If y is a certificate of infeasibility, then so is γy, $\gamma > 0$. This, however, still identifies the same variables and constraints.

An LP can have more than one certificate of infeasibility. Example.

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Continued Application of Farkas' Lemma

$$\dot{x}_1 = -2x_1^3 - 0.5x_1x_2 - 0.5x_1$$
$$\dot{x}_2 = 0.25x_1x_2^2 - 0.125x_1x_2 + 0.25x_2^2 - 0.4125x_2$$

Using $B = [-3, 3]^2$ and $d_V = 2$, the resulting LP is infeasible. Applying Farkas' Lemma once identifies the constraints

$$\mathcal{I}^1 = \{39, 40, 41, 44, 45\}.$$

Setting the corresponding rows of A equal to zero, the reduced problem is also infeasible. Applying Farkas' Lemma on the reduced problem identifies the constraints

$$\mathcal{I}^2 = \{7, 8, 11, 12, 15\}.$$

Setting the corresponding rows of A equal to zero, the second reduced problem is feasible. Thus the union

$$\mathcal{I} = \bigcup_{i=1,2} \mathcal{I}^i = \{7, 8, 11, 12, 15, 39, 40, 41, 44, 45\}$$

contains all the constraints responsible for the infeasibility of the LP.

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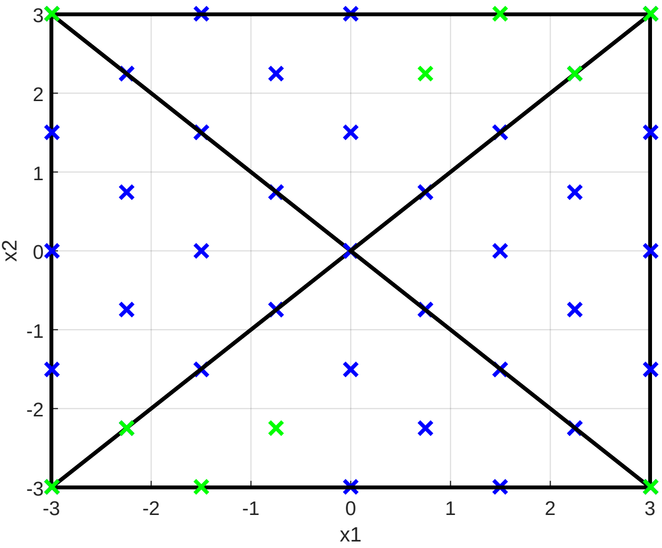
Identified Constraints



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Open Questions

- Can the infeasibility certificate be "generalised" to find all constraints at once?
- How should the information be utilized? What is the "best" way of sub-dividing?
- Is there any structure in the problem? Actually yes, the A matrix can be set on Dual Block Angular Structure. Dedicated solver?
- Benchmark Study comparing complexities with other methods.

Thank you for your attention - Questions?

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