Distributed Robustness Analysis

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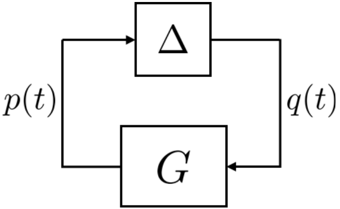
Outline

Robustness Analysis

Consider the following uncertain system,



$$p = Gq, \quad q = \Delta(p), \tag{1}$$

where $G \in \mathcal{RH}_\infty^{p \times m}$ is a transfer function matrix, and $\Delta : \mathcal{L}_2^p \to \mathcal{L}_2^m$ is a bounded and causal operator.

The uncertain system in (1) is said to be robustly stable if the interconnection between G and Δ remains stable for all Δ in some class.

Integral Quadratic Constraints

Let $\Delta : \mathcal{L}_2^p \to \mathcal{L}_2^m$ be a bounded and causal operator. This operator is said to satisfy the IQC defined by Π, i.e., $\Delta \in \text{IQC}(\Pi)$, if

$$\int_0^\infty \begin{bmatrix} v \\ \Delta(v) \end{bmatrix}^T \Pi \begin{bmatrix} v \\ \Delta(v) \end{bmatrix} dt \geq 0, \quad \forall v \in \mathcal{L}_2^p \ , \tag{2}$$

where Π is a bounded and self-adjoint operator. Assuming that Π is linear time-invariant and has a transfer function matrix representation, the IQC in (2) can be written in the frequency domain as

$$\int_{-\infty}^\infty \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix} d\omega \geq 0, \tag{3}$$

where $\hat{v}$ and $\widehat{\Delta(v)}$ are the Fourier transforms of the signals

Stability Theorem

Theorem (IQC analysis)

The uncertain system in (1) *is robustly stable, if*

1. *for all* $\tau \in [0, 1]$ *the interconnection described in* (1), *with* $\tau\Delta$, *is well-posed;*
2. *for all* $\tau \in [0, 1]$, $\tau\Delta \in \mathrm{IQC}(\Pi)$*;*
3. *there exists* $\epsilon > 0$ *such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I, \quad \forall \omega \in [0, \infty]. \tag{4}$$

Proof.

See Megretski and Rantzer, 1997. □

Example

If Δ is a linear operator, i.e. $q = \Delta p$, where $\Delta = \delta I$, $\delta \in [-1, 1]$, then

$$\Pi(j\omega) = \begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & -X(j\omega) \end{bmatrix}$$

where $X(j\omega) = X(j\omega)^* \succeq 0$ and $Y(j\omega) = -Y(j\omega)^*$.

Typically Π is parameterized with basis functions.

Collection of Uncertain Systems

Consider a collection of uncertain systems:

$$\begin{aligned} p^i &= G^i_{pq} q^i + G^i_{pw} w^i \\ z^i &= G^i_{zq} q^i + G^i_{zw} w^i \\ q^i &= \Delta^i(p^i), \end{aligned} \tag{5}$$

and let $p = (p^1, \ldots, p^N)$, $q = (q^1, \ldots, q^N)$, $w = (w^1, \ldots, w^N)$ and $z = (z^1, \ldots, z^N)$.

Interconnection of Uncertain Systems

$$\underbrace{\begin{bmatrix} w^1 \\ w^2 \\ \vdots \\ w^N \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1N} \\ \Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{N1} & \Gamma_{N2} & \cdots & \Gamma_{NN} \end{bmatrix}}_{\Gamma} \underbrace{\begin{bmatrix} z^1 \\ z^2 \\ \vdots \\ z^N \end{bmatrix}}_{z} \tag{6}$$

Each of the blocks Γ_{ij} are 0-1 matrices.

Interconnected uncertain system:

$$\begin{aligned} p &= G_{pq}q + G_{pw}w \\ z &= G_{zq}q + G_{zw}w \\ q &= \Delta(p) \\ w &= \Gamma z, \end{aligned} \tag{7}$$

where $G_{\star\bullet} = \mathrm{diag}(G^1_{\star\bullet}, \ldots, G^N_{\star\bullet})$ and $\Delta = \mathrm{diag}(\Delta^1, \ldots, \Delta^N)$.

Lumped Formulation

Eliminate w:

$$p = \bar{G}q, \quad q = \Delta(p), \tag{8}$$

where $\bar{G} = G_{pq} + G_{pw}(I - \Gamma G_{zw})^{-1}\Gamma G_{zq}$.

The interconnected uncertain system is robustly stable if there exists a matrix $\bar{\Pi}$ such that

$$\begin{bmatrix} \bar{G}(j\omega) \\ I \end{bmatrix}^* \bar{\Pi}(j\omega) \begin{bmatrix} \bar{G}(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I, \quad \forall \omega \in [0, \infty], \tag{9}$$

for some $\epsilon > 0$. LMI is *dense*.

Sparse Formulation

Theorem

Let $\Delta \in \mathrm{IQC}(\bar{\Pi})$. If there exist $\bar{\Pi}$ and $X = xI \succ 0$ such that

$$\begin{bmatrix} G_{pq} & G_{pw} \\ G_{zq} & G_{zw} \\ I & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \bar{\Pi}_{11} & 0 & \bar{\Pi}_{12} & 0 \\ 0 & -\Gamma^T X \Gamma & 0 & \Gamma^T X \\ \bar{\Pi}_{21} & 0 & \bar{\Pi}_{22} & 0 \\ 0 & X\Gamma & 0 & -X \end{bmatrix} \begin{bmatrix} G_{pq} & G_{pw} \\ G_{zq} & G_{zw} \\ I & 0 \\ 0 & I \end{bmatrix} \preceq -\epsilon I, \quad (10)$$

for $\epsilon > 0$ and for all $\omega \in [0, \infty]$, then the interconnected uncertain system in (7) is robustly stable.

Sparsity in SDPs

General SDP (new definition of x):

$$\underset{S,x}{\text{minimize}} \quad c^T x \tag{11a}$$

$$\text{subject to} \quad F^0 + \sum_{i=1}^{m} x_i F^i + S = 0, \quad S \succeq 0. \tag{11b}$$

with $S \in \mathbf{S}^n$, $x \in \mathbb{R}^m$, $c \in \mathbb{R}^m$ and $F^i \in \mathbf{S}^n$ for $i = 0, \dots, m$.

Slack variable S inherits sparsity pattern from problem data.

Solvers like DSDP (Benson and Ye, 2005) and and SMCP (Andersen, Dahl and Vandenberghe, 2010) make use of this structure.

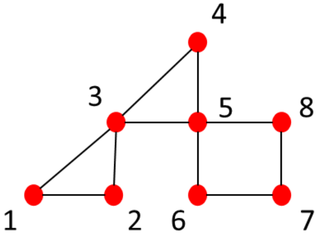
Sparsity Graph

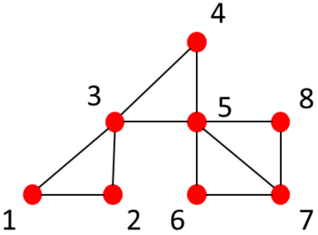
A *sparsity pattern* is a set $E \subseteq \{\{i, j\} \mid i, j \in \{1, 2, \ldots, n\}\}$.

A matrix $A \in \mathbf{S}^n$ is said to have a sparsity pattern E if $A_{i,j} = A_{j,i} = 0$, whenever $i \neq j$ and $\{i, j\} \notin E$, or equivalently $A \in \mathbf{S}^n_E$.

The graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$ is called the *sparsity graph* associated with the sparsity pattern.

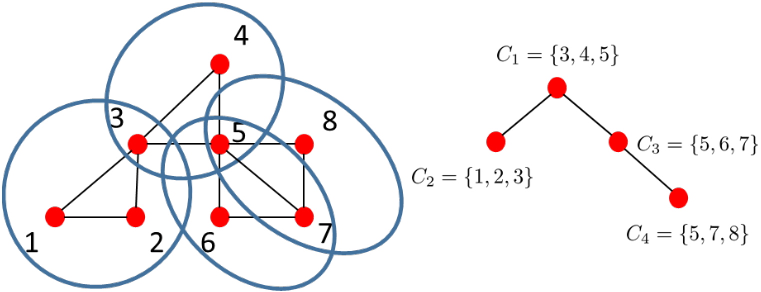
Chordal Graphs and Sparsity Patterns





$$A = \begin{bmatrix} x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & 0 & 0 & 0 & 0 & 0 \\ x & x & x & x & x & 0 & 0 & 0 \\ 0 & 0 & x & x & x & 0 & 0 & 0 \\ 0 & 0 & x & x & x & x & * & x \\ 0 & 0 & 0 & 0 & x & x & x & 0 \\ 0 & 0 & 0 & 0 & * & x & x & x \\ 0 & 0 & 0 & 0 & x & 0 & x & x \end{bmatrix}$$

Cliques and Clique Trees



A *maximal clique* C_i is a maximal subset of V such that its induced subgraph is complete.

A tree of maximal cliques for which $C_i \cap C_j$ for $i \neq j$ is contained in all the cliques on the path connecting C_i and C_j is said to have the *clique intersection property*. (Always exists.)

Sparse Cholesky Factorization

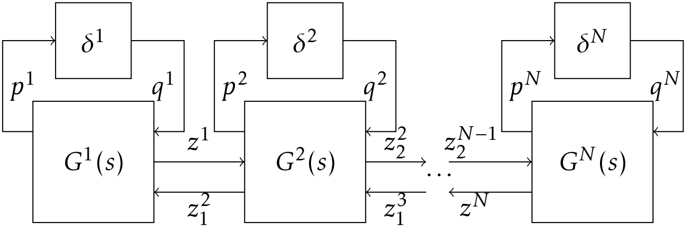
A sparsity pattern E is chordal if and only if any positive definite matrix $A \in \mathbf{S}^n_E$ has a Cholesky factorization $PAP^T = LDL^T$ with

$$P^T(L + L^T)P \in S^n_E$$

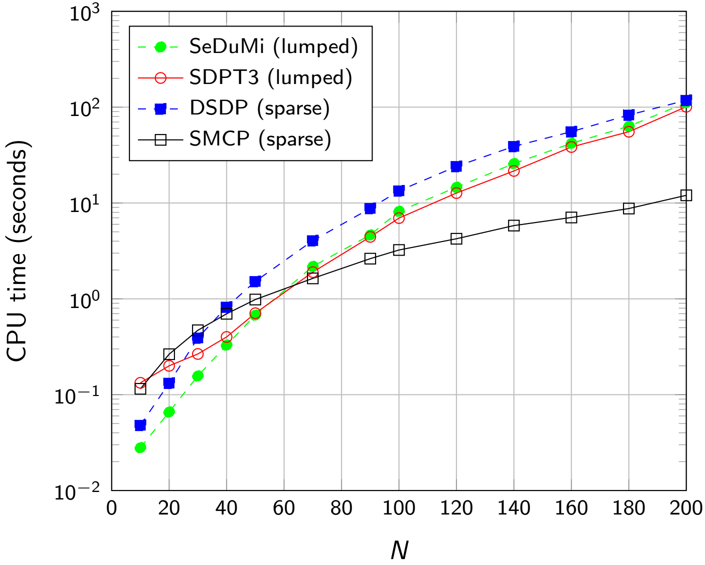
for some permutation matrix P, which is related to the clique intersection property.

After permutation sparse postive definite matrices with chordal sparsity pattern have sparse Cholesky factorizations with no fill-in.

Chain of Uncertain Systems



Average CPU Time



Test for Positive Semidefiniteness (Grone et al., 1984)

A partially specified matrix $A \in \mathbf{S}^n$ can be completed to a positive semidefinite matrix if and only if

$$A_{C_i} \succeq 0$$

where C_i are the maximal cliques of the graph for the specified entries. (A_{C_i} denotes the sub-matrices obtained by picking out the columns and rows indexed by C_i)

Example:

$$\begin{bmatrix} 1 & 1/2 & ? \\ 1/2 & 1 & 1/3 \\ ? & 1/3 & 1 \end{bmatrix} \succeq 0 \Leftrightarrow \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \succeq 0 \ \& \ \begin{bmatrix} 1 & 1/3 \\ 1/3 & 1 \end{bmatrix} \succeq 0$$

Dual SDP

Primal problem again:

$$\underset{S,x}{\text{minimize}} \quad c^T x \tag{12a}$$

$$\text{subject to} \quad F^0 + \sum_{i=1}^{m} x_i F^i + S = 0, \quad S \succeq 0. \tag{12b}$$

with chordal S with cliques C_j, $j = 1, \ldots, p$.

Dual SDP:

$$\underset{Z}{\text{minimize}} \ \operatorname{tr} ZF^0 \tag{13a}$$

$$\text{subject to} \ \operatorname{tr} ZF^i = c_i, \ \ i = 1, \ldots, m \tag{13b}$$

$$Z \succeq 0 \tag{13c}$$

Domain-Space Decomposition (Fukuda et al., 2000)

Write $F^i = \sum_{j=1}^{p} E_j F_j^i E_j^T$ with E_j containing columns of identity matrix indexed by clique C_j. (Not unique)

Since $\mathrm{tr}\, ZF^i = \sum_{j=1}^{p} \mathrm{tr}\, E_j^T Z E_j F_j^i$, equivalent dual problem is:

$$\underset{Z}{\text{minimize}} \sum_{j=1}^{p} \mathrm{tr}\, Z_{C_j} F_j^0 \tag{14a}$$

$$\text{subject to} \sum_{j=1}^{p} \mathrm{tr}\, Z_{C_j} F_j^i = c_i, \; i = 1, \ldots, m \tag{14b}$$

$$Z_{C_j} \succeq 0 \;\; i = 1, \ldots, p \tag{14c}$$

Consensus Constraints

Equivalantly in decoupeled form:

$$\underset{Z}{\text{minimize}} \sum_{j=1}^{p} \operatorname{tr} Z_j F_j^0 \tag{15a}$$

$$\text{subject to} \sum_{j=1}^{p} \operatorname{tr} Z_j F_j^i = c_i, \;\; i = 1, \ldots, m \tag{15b}$$

$$Z_j \succeq 0, \;\; j = 1, \ldots, p \tag{15c}$$

$$E_{i,j}^T \left(E_i Z_i E_i^T - E_j Z_j E_j^T \right) E_{i,j} = 0, \tag{15d}$$

$\forall\ i, j$, where i are children of j in a clique tree with the clique intersection property, and where j are all non-leaf nodes of the tree. $E_{i,j}$ contains the columns of the identity matrix indexed by $C_i \cap C_j$.

Range-Space Decomposition (Fukuda et al., 2000)

The dual of the previous problem is

$$\underset{x,U}{\text{minimize}}\ c^T x \tag{16a}$$

$$\text{subject to } F_j^0 + \sum_{i=1}^{m} x_i F_j^i + G_j(U) \succeq 0 \ j = 1, \ldots, p \tag{16b}$$

where $x \in \mathbb{R}^m$, with

$$G_j(U) = E_k^T E_{k,j} U_{k,j} E_{k,j}^T E_k - \sum_{i \in \mathrm{ch}(j)} E_j^T E_{i,j} U_{i,j} E_{i,j}^T E_j$$

where $U_{i,j} \in \mathbf{S}^{|C_i \cap C_j|}$, and where k is the parent of j in the clique tree. (For the root and for the leafs some of the terms are not there)

Often the above LMIs are loosely coupled, i.e. many F_j^i are zero.

Example

Find $x = (x_1, \ldots, x_4)$ such that

$$\begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & x_1 & x_3 \\ 0 & x_3 & x_4 \end{bmatrix} \succeq 0$$

is equivalent to find (x, u) such that

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 + u \end{bmatrix} \succeq 0 \ \& \ \begin{bmatrix} -u & x_3 \\ x_3 & x_4 \end{bmatrix} \succeq 0$$

Decomposition and Product Space Formulation

Feasibility problem from range-space decomposition can with $v = (x, U)$ be phrased as

$$\begin{aligned} \text{find} \quad & v && (17a) \\ \text{subject to} \quad & v \in \mathcal{C}_j, \quad j = 1, \ldots, p, && (17b) \end{aligned}$$

where

$$\mathcal{C}_j = \left\{ v \middle| F_j^0 + \sum_{i=1}^{m} x_i F_j^i + G_j(U) \succeq 0 \right\}$$

Let

$$\bar{\mathcal{C}}_j = \{ s^j \in \mathbb{R}^{|\mathcal{J}_j|} \mid E_{\mathcal{J}_j}^T s^j \in \mathcal{C}_i \}, \quad j = 1, \ldots, p, \qquad (18)$$

such that $s^j \in \bar{\mathcal{C}}_j$ implies $E_{\mathcal{J}_j}^T s^j \in \mathcal{C}_j$, where $E_{\mathcal{J}_j}$ are composed of rows of the identity matrix indexed by the set $\mathcal{J}_j$, which is the set of i such that v_i is constrained by $\mathcal{C}_j$. Let $\mathcal{I}_i = \{k | i \in \mathcal{J}_k\}$, i.e. the set of indeces of constraints, which depends on v_i.

Example revisited

Find $(x, u) = (x_1, x_2, x_3, x_4, u)$ such that

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 + u \end{bmatrix} \succeq 0 \ \& \ \begin{bmatrix} -u & x_3 \\ x_3 & x_4 \end{bmatrix} \succeq 0$$

Hence

$$\mathcal{J}_1 = \{1, 2, 5\}; \ \mathcal{J}_2 = \{3, 4, 5\}$$

and

$$\mathcal{I}_1 = \{1\}; \ \mathcal{I}_2 = \{1\}; \ \mathcal{I}_3 = \{2\}; \ \mathcal{I}_4 = \{2\}; \ \mathcal{I}_5 = \{1, 2\}$$

Product Space Formulation

Then (17) is equivalent to

$$\begin{aligned} \text{find} \quad & s^1, s^2, \ldots, s^p, v & (19a) \\ \text{subject to} \quad & s^i \in \bar{\mathcal{C}}_i, \quad i = 1, \ldots, p & (19b) \\ & s^i = E_{\mathcal{J}_i} v, \quad i = 1, \ldots, p & (19c) \end{aligned}$$

or

$$\begin{aligned} \text{find} \quad & S \\ \text{subject to} \quad & S \in \mathcal{C}, \ S \in \mathcal{D} \end{aligned} \qquad (20)$$

where

$$\begin{aligned} S &= (s^1, \ldots, s^p) \in \mathbb{R}^{|\mathcal{J}_1|} \times \cdots \times \mathbb{R}^{|\mathcal{J}_p|} \\ \mathcal{C} &= \bar{\mathcal{C}}_1 \times \cdots \times \bar{\mathcal{C}}_p \\ \mathcal{D} &= \{\bar{E} v \mid v \in \mathbb{R}^n\} \\ \bar{E} &= \begin{bmatrix} E_{\mathcal{J}_1}^T & \cdots & E_{\mathcal{J}_p}^T \end{bmatrix}^T. \end{aligned}$$

Example revisited

$$\text{find} \quad s^1, s^2, v \tag{21a}$$

$$\text{subject to} \quad s^1 \in \left\{ s^1 \middle| \begin{bmatrix} s^1_1 & s^1_2 \\ s^1_2 & s^1_1 + s^1_3 \end{bmatrix} \succeq 0 \right\}, \tag{21b}$$

$$s^2 \in \left\{ s^2 \middle| \begin{bmatrix} -s^2_3 & s^2_1 \\ s^2_1 & s^2_2 \end{bmatrix} \succeq 0 \right\}, \tag{21c}$$

$$s^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} v \tag{21d}$$

$$s^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} v \tag{21e}$$

Convex Minimization Formulation

Consider

$$\underset{S}{\text{minimize}} \quad F(S) := \frac{1}{2}\|S - P_{\mathcal{C}}(S)\|^2 + \frac{1}{2}\|S - P_{\mathcal{D}}(S)\|^2, \qquad (22)$$

where $P_{\mathcal{C}}(S)$ is the projection of S on the set $\mathcal{C}$ and similarly for $\mathcal{D}$.

This problem provides a solution to (20) if the optimal value is zero. A non-zero optimal value proves that (20) is infeasible.

Splitting

We equivalently write the problem with $x = S$ (new meaning of x) as

$$\underset{x,y}{\text{minimize}}\ f_1(x) + f_2(y) \tag{23a}$$

$$\text{subject to } x = y \tag{23b}$$

where

$$f_1(x) = \frac{1}{2}\|x - P_{\mathcal{C}}(x)\|^2; \quad f_2(x) = \frac{1}{2}\|x - P_{\mathcal{D}}(x)\|^2$$

Proximity Operator

Given a closed convex function $f : \mathbb{R}^n \to \mathbb{R}$, then for every $x \in \mathbb{R}^n$ the proximity operator of the function f, $\operatorname{prox}_f : \mathbb{R}^n \to \mathbb{R}^n$ is defined as the unique minimizer of the following optimization problem,

$$\underset{y}{\text{minimize}} \quad f(y) + \frac{1}{2}\|x - y\|^2.$$

Alternating Linearization Methods

Algorithm 1 ALM

1: Given $y^{(1)}$
2: **for** $k = 1, 2 \ldots$ **do**
3: $\quad x^{(k+1)} = \text{prox}_{f_1}(y^{(k)} - \nabla f_2(y^{(k)}))$
4: $\quad y^{(k+1)} = \text{prox}_{f_2}(x^{(k+1)} - \nabla f_1(x^{(k+1)}))$
5: **end for**

where

$$\text{prox}_{f_1}(x) = \frac{x + P_{\mathcal{C}}(x)}{2}; \quad \text{prox}_{f_2}(x) = \frac{x + P_{\mathcal{D}}(x)}{2}$$

and

$$\nabla f_1(x) = x - P_{\mathcal{C}}(x); \quad \nabla f_2(x) = x - P_{\mathcal{D}}(x)$$

Distributed Implementation

Since

$$(P_{\mathcal{C}}(x))_i = P_{\bar{\mathcal{C}}_i}(x^i)$$

these projections can be distributed over p computatational agents.

Moreover

$$P_{\mathcal{D}}(x) = \bar{E}\left(\bar{E}^T\bar{E}\right)^{-1}\bar{E}^T x$$

where $\bar{E}^T\bar{E} = \text{diag}(|\mathcal{I}_i|)$. Thus for the example

$$(P_{\mathcal{D}}(x))_1 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} x^1 + \begin{bmatrix} & & \\ & & \\ & & 1/2 \end{bmatrix} x^2$$

$$(P_{\mathcal{D}}(x))_2 = \begin{bmatrix} & & \\ & & \\ & & 1/2 \end{bmatrix} x^1 + \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} x^2$$

and hence this projection requires information from neighboring computational agents which have variables in common.

Scale-Free Network

- Interconection of 500 subsystems over randomly generated scale-free network, in this case a tree.
  - 478 systems connected to 5 or less other systems
  - 16 systems connected to less than 11 but more than 5 other systems
  - 6 system connected to more than 10 other systems

Scale-Free Network ctd.

- Lumped formulation: LMI of dimension 500 with 500 variables
- Sparse formulation: LMI of dimension 1498 with 1498 variables
- Chordal embedding has 579 cliques with 9894 variables
- Largest LMI has dimension 210 and 170 variables, but 94% of them has dimension 50 or less
- The largest coupling between LMIs involves 92 variables, but 95% of them involve less than 24 variables.
- One of the agents require information from 52 other agents, but 96 % of the agents only require information from at most 10 other agents.

Numerical Results

Solver	Avg. CPU time [sec]
SDPT3 (lumped)	5640
SeDuMi (lumped)	2760
DSDP (sparse)	167
SMCP (sparse)	33
ALM (sparse)	1623

- ALM only prototyped in Matlab
- ALM can use paralell processors
- ALM respect privacy

Domain-Space Decomposition Revisited

Another equivalent formulation of the dual problem is:

$$\underset{Z, Z_j}{\text{minimize}} \sum_{j=1}^{p} \operatorname{tr} Z_j F_j^0 \tag{24a}$$

$$\text{subject to} \sum_{j=1}^{p} \operatorname{tr} Z_j F_j^i = c_i, \; i = 1, \ldots, m \tag{24b}$$

$$Z_j \succeq 0, \; j = 1, \ldots, p \tag{24c}$$

$$E_j^T Z E_j = Z_j, \; j = 1, \ldots, p \tag{24d}$$

where we now have many more variables due to the additional variable Z.

Remember that many F_j^i are zero.

Search Directions for Interior-Point (IP) Methods

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta\bar{z} \\ \Delta\bar{x} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

where H and A are sparse. ($\bar{z}$ vector of all elements of Z and Z_i, $i = 1, 2, \ldots, p$)

Equivalently the optimiality conditions of

$$\underset{\Delta\bar{z}}{\text{minimize}} \ \frac{1}{2}\Delta\bar{z}^T H \Delta\bar{z} - r_1^T \Delta\bar{z} \tag{25a}$$

$$\text{subject to } A\,\Delta\bar{z} = r_2 \tag{25b}$$

After Elimination of the Z_j-variables

$$\underset{\Delta\tilde{z}}{\text{minimize}}\ \frac{1}{2}\Delta\tilde{z}^T\tilde{H}\Delta\tilde{z} - \tilde{r}_1^T\Delta\tilde{z} \tag{26a}$$

$$\text{subject to}\ \tilde{A}\,\Delta\tilde{z} = \tilde{r}_2 \tag{26b}$$

which still has sparse data matrices.

Allmost separable

$$\underset{\Delta z_i}{\text{minimize}}\ \frac{1}{2}\sum_{i=1}^{p}\Delta\tilde{z}_i^T\tilde{H}_i\Delta\tilde{z}_i - \tilde{r}_{1,i}^T\Delta\tilde{z}_i \tag{27a}$$

$$\text{subject to}\ \sum_{j\in\tilde{\mathcal{J}}_i}\tilde{A}_{i,j}\,\Delta\tilde{z}_j = \tilde{r}_{2,i}, \quad i = 1, 2, \ldots, m \tag{27b}$$

where $\tilde{\mathcal{J}}_i$ are small subsets of $\{1, 2, \ldots, p\}$.

Equivalent unconstrained problem

$$\underset{\Delta\tilde{z}}{\text{minimize}} \sum_{i=1}^{p} F_i(\Delta\tilde{z})$$

where

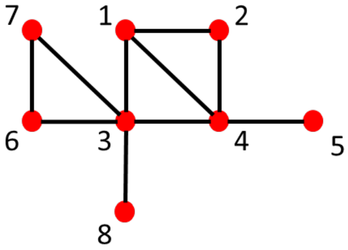
$$F_i(\Delta\tilde{z}) = \frac{1}{2}\Delta\tilde{z}_i^T \tilde{H}_i \Delta\tilde{z}_i - \tilde{r}_{1,i}^T \Delta\tilde{z}_i + I_{\mathcal{D}_i}(\Delta\tilde{z})$$

with $I_{\mathcal{D}_i}$ the indicator function for the set described by the ith equality constraint.

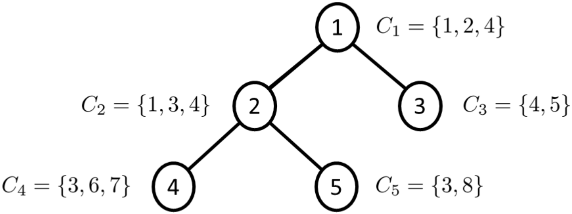
Simple Example

$$\underset{x}{\text{minimize}} \quad \bar{F}_1(x_1, x_3) + \bar{F}_2(x_1, x_2, x_4) + \bar{F}_3(x_4, x_5) + \bar{F}_4(x_3, x_4) + \bar{F}_5(x_3, x_6, x_7) + \bar{F}_6(x_3, x_8). \quad (28)$$

Has sparsity graph (edge between vertices if components in same term)



Clique Tree for Sparsity Graph



We now assign one computational agent for each clique, and we may assign $\bar{F}_i$ to an agent if and only if the indeces of its variables belong to the corresponding clique. Hence we can assign $\bar{F}_1 + \bar{F}_4$ to C_2, $\bar{F}_2$ to C_1, $\bar{F}_3$ to C_3, $\bar{F}_5$ to C_4 and $\bar{F}_6$ to C_5. (Not unique assignment)

Message Passing or Dynamic Programming over Trees

Start with the leaves and compute for agents 3, 4, and 5

$$m_{31}(x_4) = \min_{x_5} \left\{ \bar{F}_3(x_4, x_5) \right\} \tag{29}$$

$$m_{42}(x_3) = \min_{x_6, x_7} \left\{ \bar{F}_5(x_3, x_6, x_7) \right\} \tag{30}$$

$$m_{52}(x_3) = \min_{x_8} \left\{ \bar{F}_6(x_3, x_8) \right\} \tag{31}$$

Then add the results from agents 4 and 5 to the functions of Agent 2 and compute

$$m_{21}(x_1, x_4) = \min_{x_3} \left\{ \bar{F}_1(x_1, x_3) + \bar{F}_4(x_3, x_4) + m_{42}(x_3) + m_{52}(x_3) \right\} \tag{32}$$

Finally add the results from agents 2 and 3 to the functions of Agent 1 and compute

$$\min_{x_1, x_2, x_4} \left\{ \bar{F}_2(x_1, x_2, x_4) + m_{31}(x_4) + m_{21}(x_1, x_4) \right\}$$

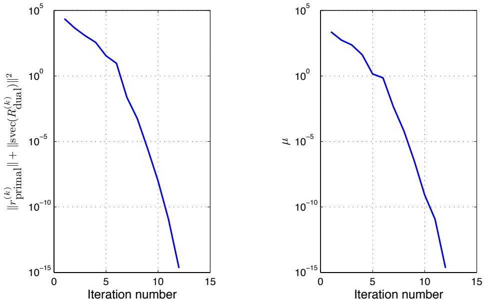
Comments

- Not easy in general to compute messages or value functions $m_{i,j}$.
- For linearly constrained convex quadratic problems the messages are convex quadratic functions with equality constraints.
- The dual variables can also be recovered.
- In fact results in a *multi-frontal factorization technique* for the KKT saddle point problem.

Comments ctd.

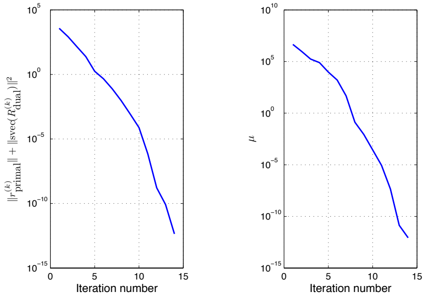
- The cliques for the search directions of the dual problem obtained using domain-space decomposition will not be the same as the cliques in the domain-space decomposition itself.
- They can however be obtained by “merging” cliques, where one clique might have to be merged with several others.
- All other computations in an IP algorithm also distribute over the clique tree.
- In total 6 upward and 6 downward passes through the clique tree, of which only one pass involves significant computations, for each iteration in an IP algorithm

Chain of 100 Uncertain Systems



- 198 cliques
- Height of qlique tree 99
- Largest clique of dimension 8.
- Each agent computed a factorization 12 times and needed to communicate with its neighbours 144 times.
- Dimension of marix to factorize was at most 62.
- Each agent had at most 2 neighbours.

The Scale-Free Network



- 579 cliques
- Height of qlique tree 35
- Largest clique of dimension 162.
- Each agent computed a factorization 14 times and needed to communicate with its neighbours 168 times.
- Dimension of marix to factorize was at most 5456.
- Each agent had at most 39 neighbours.

Summary

- Presented *scalable distributed* optimization algorithms that respect *privacy*.
- However, distributed solutions more costly when implemented centralized and especially so for second order methods.
- Robustness analysis has applications in power grids.
- Distributed localization of scattered sensor networks.
- Distributed predictiv control of platoons of vehicles.
- Distributed inertial motion capture

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